Tracial Approximation for crossed products by finite groups with the tracial Rokhlin property

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Tracial approximation

Let TAC be the class of C*-algebras which can be *tracially approximated* by C*-algebras in the class $\mathcal{C}$ in the sense of Elliott and Niu: for any $\varepsilon > 0$, any finite set $\mathcal{F} \subset A$, and any non-zero $a \in A^+$, there exist a non-zero projection $p \in A$ and a sub-C*-algebra $C \subset A$ such that $C \in \mathcal{C}$, $1_C = p$, and for all $x \in \mathcal{F}$,

1. $\|xp - px\| < \varepsilon$

2. $pxp \in_\varepsilon C$, and

3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

- (Lin 01) When $\mathcal{C}$ is the set of all finite dimensional C*-algebras, then a C*-algebra $A \in TA\mathcal{C}$ is called a traicial approximated finite dimensional algebra, i.e., TAF-C*-algebras.

- (Lin 04) When $\mathcal{C}$ is the set of all interval algebras, that is, C*-algebras isomorphic to $F \otimes C[0,1]$ for a finite dimensional algebra F, then a C*-algebra $A \in TA\mathcal{C}$ is called a traicial approximated interval algebra, i.e., TAI-C*-algebras.

Note that the classification theorems for TAF-algebras and TAI-algebras were given by Lin in 01 and 04, respectively.

In this talk we show that for any simple C*-algebra $A \in TA\mathcal{C}$ an an action α of a finite group G if α has the tracial Rokhlin property, then the crossed product algebra $C^*(G, A, \alpha)$ belongs to the class $TA\mathcal{C}$.

Rokhlin property

The Rokhlin property for finite group actions is formulated by Izumi as follows;

Definition 1 (Izumi 04). Let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *strict Rokhlin property* if for every finite set $F \subset A$, and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.

2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.

3. $\sum_{g \in G} e_g = 1$.

The tracial Rokhlin property weakens Condition (3) of this definition:

Definition 2 (Phillips 06). Let A be an infinite dimensional simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.

2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.

3. With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

4. With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

When A is finite, we do not need Condition (4) of Definition 2:

Example 3. Let $\mathbb{M}_{n^\infty} = \otimes_{k=1}^{\infty} \mathbb{M}_n(\mathbf{C})$ and

$$\alpha = \otimes_{k=1}^{\infty} \mathrm{Ad} \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \cdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $\{\lambda_i\}_{i=1}^{n}$ is the root of the unit. Then α be the automorphism of order n on $\mathbb{M}_{n^\infty}$, and α has the Rokhlin property. $\square$

We also construct an action which does not have the Rokhlin property.

Proposition 4 (Phillips 06). Let D be an infinite tensor product C*-algebra and let $\alpha \in \mathrm{Aut}(D)$ be an automorphism of oreder 2, of the form

$$D = \otimes_{n=1}^{\infty} \mathbb{M}_{k(n)}(\mathbf{C}) \text{ and } \alpha = \otimes_{n=1}^{\infty} \mathrm{Ad}(p_n - q_n),$$

with $k(n) \in \mathbf{N}$ and where $p_n, q_n \in \mathbb{M}_{k(n)}(\mathbf{C})$ are projections with $p_n + q_n = 1$ and $\mathrm{rank}(p_n) \geq \mathrm{rank}(q_n)$ for all $n \in \mathbf{N}$. Set

$$\lambda_n = \frac{\mathrm{rank}(p_n) - \mathrm{rank}(q_n)}{\mathrm{rank}(p_n) + \mathrm{rank}(q_n)}$$

for $n \in \mathbf{N}$ and, for $m \leq n$ $\Lambda(m, n) = \lambda_{m+1}\lambda_{m+2} \cdots \lambda_n$ and $\Lambda(m, \infty) = \lim_{n \to \infty} \Lambda(m, n)$. Then the followings are equivalent:

(1) The action α has the Roklin property.

(2) There are infinitely many $n \in \mathbf{N}$ such that $\mathrm{rank}(p_n) = \mathrm{rank}(q_n)$, i.e. $\lambda_n = 0$.

(3) $C^*(\mathbf{Z}_2, D, \alpha)$ is a UHF algebra.

□

Proposition 5 (Phillips 06). Let $\alpha \in \mathrm{Aut}(D)$ be a product type automorphism of order 2 as in Proposition 4. Then the followings are equivalent:

(1) The action α has the tracial Rokhlin property.

(2) $\Lambda(m, \infty) = 0$ for all m.

□

Basic Lemma

Lemma 6 (Phillips 06, Archey 09). Let a be an infinite dimensional stably finite simple C*-algebra with the Property SP. Let G be a finite group of order n and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G with the tracial Rokhlin property. Then for any $\varepsilon > 0$, every $N \in \mathbb{N}$, any finite set $\mathcal{F} \subset C^*(G, A, \alpha)$, and any non-zero $z \in (C^*(G, A, \alpha))^+$, there exist a non-zero projection $e \in A \subset A \rtimes_\alpha G$, a unital C*-subalgebra $D \subset C^*(G, A, \alpha)$, a projection $f \in A$ and an isomorphism $\phi\colon M_n \otimes fAf \to D$, such that

- For every $a \in \mathcal{F}$ $\|ea - ae\| < n\varepsilon$.
- With (e_{gh}) for $g, h \in G$ being a system of matrix units for M_n, we have $\phi(e_{11} \otimes a) = a$ for all $a \in fAf$ and $\phi(e_{gg} \otimes 1) \in A$ for $g \in G$.
- With (e_{gg}) as in (1), we have $\|\phi(e_{gg} \otimes a) - \alpha_g(a)\| \leq \varepsilon \|a\|$ for all $a \in fAf$.
- For every $a \in F$ there exist $b_1, b_2 \in D$ such that $\|ea - b_1\| < \varepsilon$, $\|ae - b_2\| < \varepsilon$ and $\|b_1\|, \|b_2\| \leq \|a\|$.
- $e = \sum_{g \in G} \phi(e_{gg} \otimes 1)$.

- $1-e$ is Murray-von Neumann equivalent to a projection in $\overline{z(C^*(G,A,\alpha))z}$.

Here a C*-algebra A is said to have *the Property SP* if every nonzero hereditary subalgebra of A has nonzero projection.

Main Theorem

Theorem 7. Let $\mathcal{C}$ be a class of infinite dimensional stably finite separable unital C*-algebras which is closed under the follwoing conditions:

(1) $A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$.

(2) If $A \in \mathcal{C}$ and $n \in \mathbf{N}$, then $M_n(A) \in \mathcal{C}$.

(3) If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.

For any simple C*-algebra $A \in TA\mathcal{C}$ and an action α of a finite group G if α has the tracial Rokhlin property, then the crossed product algebra $C^*(G, A, \alpha)$ belongs to the class $TA\mathcal{C}$.

Sketch of the proof

1. (Phillips 06) Since α has the tracial Rokhlin property, A has the property SP or α has the strictly Rokhlin property.

2. Case 1: α has the strictly Rokhlin property: Then $C^*(G, A, \alpha) \in TAC$ by [Osaka-Phillips 07].

3. Case 2: A has the property SP: There exists a non-zero projection $q \in A$ which is Murray-von Neumann equivalent in $C^*(G, A, \alpha)$ to a projection in $\overline{z(C^*(G, A, \alpha))z}$ by [Osaka 01].

   Since A is simple, take orthogonal projections q_1, q_2 with $q_1, q_2 \leq q$ by the standard argument.

   Apply Basic Lemma. Do some standard argument in the tracial topological rank theory.

Theorem 8. Let A be an infinite dimensional simple separable unital C*-algebra with stable rank one and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G with tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.

This sharpens result by [Archey 09] a little. In her paper she puts two more conditions

1. A has real rank zero.

2. The order on projections over A is determined by traces, i.e., whenever $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \preceq q$.

The proof is done by using Theorem 7 and the following result by Elliott-Niu:

Theorem 9 (Elliott-Niu 08). Let $\mathcal{C}$ be a class of unital C*-algebras with stable rank one. Then any simple C*-algebra in the class $TA\mathcal{C}$ has stable rank one.

Tracial topological rank

Definition 10. Let $\mathcal{T}^{(0)}$ be the calss of all finite dimensional C*-algebras and let $\mathcal{T}^{(k)}$ be the class of all C*-algebras with the form $pM_n(C(X))p_n$, where X is a finite CW complex with dimension k and $p \in M_n(C(X))$ is a projection.

A simple unital C*-algebra A is said to have tracial topological rank no more than k if for any set $\mathcal{F} \subset A$, and $\varepsilon > 0$ and any nonzero positive element $a \in A$, there exists a C*-subalgebra $B \subset A$ with $B \in \mathcal{T}^{(k)}$ and $id_B = p$ such that

(1) $\|xp - px\| < \varepsilon$

(2) $pxp \in_\varepsilon C$, and

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Theorem 11 (Osaka-Phillips 07). Let A be an infinite dimensional simple unital C*-algebra with tracial topological rank more than or equal to k, and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a finite group G with tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial topological rank more that or equal to k.

Proof. Let $\mathcal{C}$ be the set $\mathcal{T}^{(k)}$. Then $\mathcal{T}^{(k)}$ is closed under three conditions in Theorem 7. Then from Theorem 7 $C^*(G, A, \alpha)$ belongs to the class $TA\mathcal{C}$. This means that $C^*(G, A, \alpha)$ has tracial topological rank more that or equal to k from the Definition 10. □

Further topic

In this section we discuss about Rokhlin property for integer $\mathbf{Z}$.

Definition 12. Let A be a simple unital C*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say α has *the tracial cyclic Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbf{N}$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n$, where $e_{n+1} = e_0$.

(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n$ and for all $a \in F$.

(3) With $e = \sum_{j=0}^{n} e_j$, $1 - e$ is Murray-von Neumann equivalent to a nonzero projection in $x(C^*(\mathbf{Z}, A, \alpha)x$.

Note that when a simple unital separable C*-algebra A has tracial topological rank zero, an automorphism with the Rokhlin property in the sense of Kishimoto, or with the tracial Rokhlin property in the sense of Osaka-Phillips, has the tracial cyclic Rokhlin property (Lin-Osaka 04). We can also construct an automorphism α on a simple unital AT-algebra with the tracial cyclic Rokhlin property from an isomorphism $\gamma_1 \colon K_1(A) \to K_1(A)$ such that $\alpha_* = \gamma_1$ (Lin-Osaka 06).

The following is a Key Lemma for characterization of crossed products by $\mathbf{Z}$.

Proposition 13. Let A be a simple unital C*-algebra with Property SP. Suppose that $\alpha \in Aut(A)$ has the tracial cyclic Rokhlin property. Then for any $\varepsilon > 0$, a finite set $F \subset C^*(\mathbf{Z}, A, \alpha)$, and a nonzero positive element $z \in A \rtimes_\alpha G$, there exist a projection $e \in A \subset A \rtimes_\alpha \mathbf{Z}$, a unital subslgebra $D \subset eC^*(\mathbf{Z}, A, \alpha)e$, and a projection $f \in A$, and an isomorphism $\phi\colon M_2 \otimes fAf \to D$ such that

(1) With matrix units (e_{ij}) for M_2, we have $\phi(e_{11} \otimes a) = a$ for $a \in fAf$ and $\phi(e_{ii} \otimes e) \in A$ for $i = 1, 2$.

(2) $\|\phi(e_{22} \otimes a) - \alpha(a)\| < \epsilon \|a\|$ for all $a \in fAf$.

(3) $e = \sum_{i=1}^{2} \phi(e_{ii} \otimes f)$.

(4) $eae \subset_\varepsilon D$ for all $a \in F$.

(5) $1 - e$ is Murray-von Neumann equivalent in $A \rtimes_\alpha G$ a projection in $\overline{zC^*(\mathbf{Z}, A, \alpha)z}$.

Theorem 14. Let $\mathcal{C}$ be a class of infinite dimensional stably finite separable unital C*-algebras which is closed under the follwoing conditions:

(1) $A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$.

(2) If $A \in \mathcal{C}$ and $n \in \mathbf{N}$, then $M_n(A) \in \mathcal{C}$.

(3) If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.

For any simple C*-algebra $A \in TA\mathcal{C}$ with Property SP if an automorphism α has the tracial cyclic Rokhlin property, then the crossed product algebra $C^*(\mathbf{Z}, A, \alpha)$ belongs to the class $TA\mathcal{C}$.

Corollary 15. Let A be an infinite dimensional simple separable unital C*-algebra with stable rank one and Property SP, and let $\alpha \in \mathrm{Aut}(A)$ be an automorphism with tracial cyclic Rokhlin property. Then $C^*(\mathbf{Z}, A, \alpha)$ has stable rank one.

Theorem 16. Let A be an infinite dimensional simple unital C*-algebra with tracial topological rank more than or equal to k, and $\alpha \in \mathrm{Aut}(A)$ is an automorphism with tracial cyclic Rokhlin property. Then $C^*(\mathbf{Z}, A, \alpha)$ has tracial topological rank more that or equal to k.

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